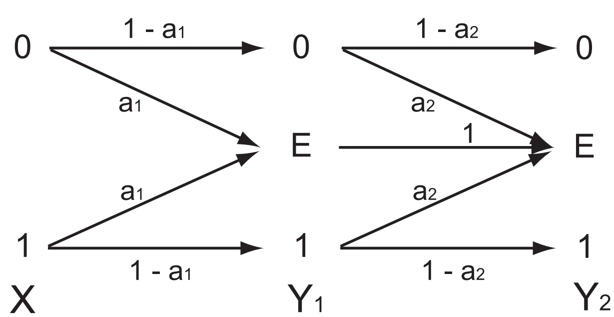
Homework 3
Due: February 28, 2011

1. What is the capacity of the following multiple access channel?

$$X_1 \in \{-1, 0, 1\}$$
$$X_2 \in \{-1, 0, 1\}$$
$$Y = X_1^2 + X_2^2$$

   (a) Find the capacity region.
   (b) Describe $p^*(x_1)$, $p^*(x_2)$ achieving a point on the boundary of the capacity region.
   (c) What is the capacity if $Y = X_1 X_2$?

2. Consider the following degraded broadcast channel.



   (a) What is the capacity of the channel from X to Y_1?
   (b) What is the channel capacity from X to Y_2?
   (c) What is the capacity region of all (R_1, R_2) achievable for this broadcast channel? Simplify and sketch.

3. Assume that a sender X is sending to two fixed base stations. Assume that the sender sends a signal X that is constrained to have average power P. Assume that the two base stations receive signals Y_1 and Y_2, where

$$Y_1 = \alpha_1 X + Z_1$$
$$Y_2 = \alpha_2 X + Z_2$$

   where $Z_1 \sim \mathcal{N}(0, N_1)$, $Z_2 \sim \mathcal{N}(0, N_2)$, and Z_1 and Z_2 are independent. We will assume the α's are constant over a transmitted block.

   (a) Assuming that both signals Y_1 and Y_2 are available at a common decoder $Y = (Y_1, Y_2)$, what is the capacity of the channel from the sender to the common receiver?
   (b) If, instead, the two receivers Y_1 and Y_2 each decode their signals independently, this becomes a broadcast channel. Let R_1 be the rate to base station 1 and R_2 be the rate to base station 2. Find the capacity region of this channel.

4. For the degraded broadcast channel $X \to Y_1 \to Y_2$, find the points a and b where the capacity region hits the R_1 and R_2 axes. Show that $b \leq a$

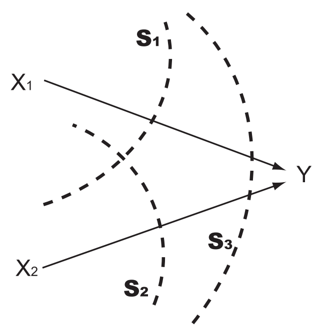
5. For the multiple access channel we know that (R_1, R_2) is achievable if

$$R_1 < I(X_1; Y|X_2)$$
$$R_2 < I(X_2; Y|X_1)$$
$$R_1 + R_2 < I(X_1, X_2; Y)$$

for X_1, X_2 independent. Show, for X_1, X_2 independent, that

$$I(X_1; Y|X_2) = I(X_1; Y, X_2)$$

Thus R_1 is less than the mutual information between X_1 and everything else.



Interpret the information bounds as bounds on the rate of flow across cutsets S_1, S_2 and S_3.

6. **Multiple-antenna Gaussian MAC.** Consider a two-user Gaussian multiple access channel with channel output $Y = (Y_1, Y_2)$ given by

$$Y_1 = X_1 + Z_1$$
$$Y_2 = X_1 + X_2 + Z_2,$$

where channel inputs X_1 and X_2 from each user are both subject to power constraint P, and the zero-mean unit-variance Gaussian noises Z_1 and Z_2 are independent of each other and channel inputs.

(a) Find the capacity region.

(b) Find the time-division region with power control. Is it possible to achieve any point on the boundary of the capacity region (except for the end points) using time-division?

7. **Duality between Gaussian BC and MAC.** Consider the following Gaussian broadcast and multiple access channels:

The broadcast channel: At time i

$$Y_{1i} = g_1 X_i + Z_{1i}$$
$$Y_{2i} = g_2 X_i + Z_{2i},$$

where the Z_{1i} and Z_{2i} are *i.i.d.* and independent of X_i, $Z_{ki} \sim \mathcal{N}(0, 1)$ for $k = 1, 2$. Assume average power constraint P on the codeword X^n.

The multiple access channel: At time i

$$Y_i = g_1 X_{1i} + g_2 X_{2i} + Z_i,$$

where the Z_i are *i.i.d.* and independent of X_i, $Z_i \sim \mathcal{N}(0,1)$. Assume a sum average power constraint on each pair of codewords $(x^n(m1), x^n(m2))$ as

$$\frac{1}{n}\sum_{i=1}^{n} x_{1i}^2 + x_{2i}^2 \leq P$$

(a) Provide expressions for the independent message capacity regions of these two channels in terms of $C(x) = \frac{1}{2}\log(1+x)$.

(b) Show that the two capacity regions are equal.

(c) Consider a point (R_1, R_2) on the boundary of the capacity region for the broadcast channel. Show that the same point exists on the boundary of the capacity for the multiple access channel with some power allocation between the two transmitters.
Using superposition coding for the broadcast channel that achieves a point on the boundary of its capacity region, argue that the same sequences of code can be used to achieve the same point on the capacity region of the given multiple access channel. What is special about the encoding order at the broadcast channel and the decoding order at the multiple access channel, assuming successive cancellation?

Notes: The above duality result is a simple case of a general duality result between Gaussian broadcast and multiple access channels established in N. Jindal, S. Vishwanath, A. J. Goldsmith, "On the duality of Gaussian multiple-access and broadcast channels," IEEE Trans. Inf. Theory, May 2004.

8. **Converse proof for the Gaussian BC.** Consider a Gaussian broadcast channel with $N_2 > N_1$. In the proof of the converse of the capacity theorem, Fano's inequality can be used to show that

$$nR_1 \leq I(M_1; Y_1^n | M_2) + n\epsilon_n$$
$$nR_2 \leq I(M_2; Y_2^n) + n\epsilon_n.$$

Now show that there exists an $\alpha \in [0,1]$ such that

$$I(M_2; Y_2^n) \leq nC\left(\frac{\bar{\alpha}P}{\alpha P + N_2}\right)$$
$$I(M_1; Y_1^n | M_2) \leq nC\left(\frac{\alpha P}{N_1}\right)$$

Hint: To prove the second inequality, first show that

$$I(M_1; Y_1^n | M_2) \leq h(Y_1^n | M_2) - \frac{n}{2}\log(2\pi e N_1),$$

then using a conditional version of the vector EPI given below, find an upper bound for $h(Y_1^n | M_2)$.

$$e^{\frac{2}{n}h((X^n+Y^n)|U)} \geq e^{\frac{2}{n}h(X^n|U))} + e^{\frac{2}{n}h(Y^n|U)}.$$

9. **Capacity of Gaussian MAC with rate splitting.** The capacity of a multiple access channel can be achieved by successive cancellation decoding to achieve the corner points of the capacity region, and time-sharing to achieve the rate points in between. Instead of time-sharing, another technique that can achieve the capacity of the Gaussian MAC is *rate splitting* at one of the encoder, and then use successive single-user decoding. The idea is to split one encoder (say encoder 2) into two encoders operating at the respective rates R_{21} and R_{22}, where $R_2 = R_{21} + R_{22}$. Suppose that these encoders generate codewords with respective powers P_{21} and P_{22}, where $P_2 = P_{22} + P_{22}$. Then the transmitted codeword from user 2 is the sum of the two codewords generated by these two encoders. The decoder performs single-user decoding in three stages: first, decode the R_{21} code; second, decode the R_1 code; third, decode the R_{22} code.

(a) Establish the achievable rate region for the Gaussian MAC using this rate-splitting and single-user decoding technique.

(b) Show that this rate region equals to the known MAC capacity region.

Notes: This rate-splitting approach to the Gaussian MAC was established in Bixio Rimoldi and Rudiger Urbanke, "A Rate-Splitting Approach to the Gaussian Multiple-Access Channel" IEEE Trans. on Info. Theory, Vol. 46, NO. 2, 1996.

10. **The Blackwell broadcast channel.** The Blackwell broadcast channel is defined as follows.

$$\mathcal{X} = \{0,1,2\}, \mathcal{Y}_1 = \mathcal{Y}_2 = \{0,1\},$$
$$p(0,0|0) = p(0,1|1) = p(1,1|2) = 1.$$

(a) Is this channel degraded? Why or why not?

(b) Propose an achievable rate region for private messages using:

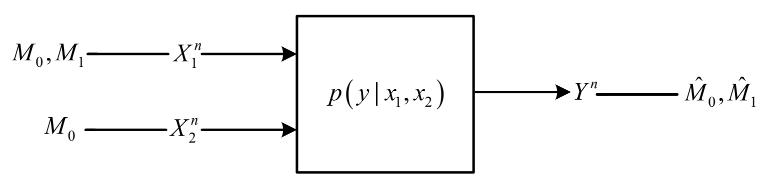
   i. Superposition coding

   ii. Marton's inner bound (over binning)

(c) Find the Sato outer bound for this channel.

(d) Plot all three bounds on a graph.

11. **Multiple access channel with degraded message sets.** Consider a DM-MAC $(\mathcal{X}_1\mathcal{X}_2, p(y|x_1,x_2), \mathcal{Y})$. Sender 1 encodes two independent messages (M_0, M_1) uniformly distributed over $[2^{nR_0}] \times [2^{nR_1}]$, while Sender 2 encodes the message M_0 only. Thus, the common message M_0 is available to both senders, while the private message M_1 is available only to Sender 1. The receiver Y needs to decode both M_0 and M_1.



Multiple access channel with DMS.

The capacity region of this channel is given by the convex closure of all rate pairs (R_1, R_2) satisfying

$$R_1 < I(X_1; Y|X_2)$$
$$R_1 + R_0 < I(X_1, X_2; Y)$$

for some $p(x_1,x_2)p(y|x_1,x_2)$.

(a) Prove the achievability of the capacity region.

(b) Prove the weak converse of the capacity region.

(c) What is the capacity region for the AWGN-MAC with degraded message sets under noise power 1 and input power constraints P_1 and P_2?

12. **More capable broadcast channel.** Let $(\mathcal{X}, p(y_1,y_2|x), \mathcal{Y}_1, \mathcal{Y}_2)$ be a DM-BC. Receiver Y_1 is said to be *more capable* than receiver Y_2 if $I(X;Y_1) \geq I(X;Y_2)$ for all input pmf $p(x)$. For the more capable broadcast channel, the capacity region is known and is achieved by superposition coding.

(a) Show that if a broadcast channel is degraded then it is more capable, but not the other way around. Hence the more capable broadcast channel is a broader class than the degraded broadcast channel.

(b) Show that if Y_1 is more capable than Y_2, then

$$I(X_n; Y_1^n) \geq I(X_n; Y_2^n)$$

for all $p(x^n)$ and for all $n \geq 1$. (Hint: Assume without loss of generality that Y_1 and Y_2 are conditionally independent given X, and use induction.)

(c) Specify the private-message rate region achievable by superposition coding for the more capable broadcast channel. Prove that it is also the capacity region.