

Lecture 4: Gaussian Channels

1 Scalar Gaussian channel

Consider a discrete-time channel in which the input X_i and output Y_i are samples at time i and are continuous signals. The noise samples Z_i are i.i.d Gaussian, and X_i and Z_i are independent. At each sample, the channel can be expressed as

$$Y_i = X_i + Z_i$$

Without any constraint, the capacity is infinity.

Average power constraint

For each codeword $X^n = (X_1, X_2, \dots, X_n)$

$$\frac{1}{n} \sum_{i=1}^n X_i^2 \leq P.$$

We can also write $E[X^2] \leq P$ although this is slightly different as it implies the average power per column of the codebook (at each time sample).

Then using the result from the DMC

$$\begin{aligned} I(X; Y) &= h(Y) - h(Y|X) = h(Y) - h(X + Z|X) \\ &= h(Y) - h(Z|X) = h(Y) - h(Z) \\ &= h(Y) - \frac{1}{2} \log 2\pi e N. \end{aligned}$$

Since $E[X^2] \leq P$ and $E[Z^2] \leq N$, so $Y = X + Z \rightarrow E(Y^2) \leq P + N$ and

$$h(Y) \leq \frac{1}{2} \log 2\pi e(P + N)$$

where the equality holds iff $Y \sim \mathcal{N}(0, P + N)$ and $X \sim \mathcal{N}(0, P)$.

Theorem

The capacity of a Gaussian channel with input power constraint P and noise variance $\sigma^2 = N$ is

$$C = \frac{1}{2} \log \left(1 + \frac{P}{N} \right).$$

This capacity is achieved with $X \sim \mathcal{N}(0, P)$.

The proof of this theorem is similar to the proof of the DMC capacity but involves extra steps concerned the power constraint.

Set the SNR = $\frac{P}{N} = \gamma$, so it could be written as $C = \frac{1}{2} \log(1 + \gamma)$. In a graph we usually plot the capacity against $\gamma_{dB} = 10 \log_{10} \gamma$.

2 Parallel Gaussian channels

Consider K parallel Gaussian channels with independent noises $Z_i \sim \mathcal{N}(0, N_i)$.

$$Y_k = X_k + Z_k, \quad k = 1 \dots K$$

The inputs X_i are independent with Z_j and have sum power constraint as

$$\sum_{i=1}^K E[X_k^2] = P.$$

Because of independent noises, the transmission rate can be upper-bounded as

$$\begin{aligned} I(X_1, X_2, \dots, X_k; Y_1, Y_2, \dots, Y_k) &\leq \sum_{i=1}^k I(X_i; Y_i) \\ &\leq \frac{1}{2} \sum_{i=1}^k \log \left(1 + \frac{P_i}{N_i} \right). \end{aligned}$$

Question: How to optimize power allocation?

$$\begin{aligned} \max \quad & \frac{1}{2} \sum \log \left(1 + \frac{P_i}{N_i} \right) \\ \text{s.t.} \quad & \sum P_i = P, \\ & P_i \geq 0 \end{aligned}$$

Form the Lagrangian, we have

$$\begin{aligned} J &= \frac{1}{2} \sum \log \left(1 + \frac{P_i}{N_i} \right) + \lambda \sum P_i \\ \frac{\partial J}{\partial P_i} &= \frac{1}{2} \frac{1}{P_i + N_i} + \lambda = 0 \longrightarrow P_i = -\frac{1}{2\lambda} - N_i. \end{aligned}$$

The optimal power allocation is

$$P_i = (\mu - N_i)^+$$

where μ is such that $\sum P_i = P$. This is called water-filling.

3 Parallel Gaussian channels with colored noise

In the case of colored noise, $(Z_1, Z_2, \dots, Z_k) \sim \mathcal{N}(0, K_z)$. (If $K_z = \text{diag}(N_1, N_1, \dots, N_k)$ then again we have independent noises as the previous case.)

The optimal input $X = (X_1, X_2, \dots, X_k)$ is $X \sim \mathcal{N}(0, K_x)$. We want to find the optimal K_x , subject to

$$\begin{aligned} E(X_1^2) + E(X_2^2) + \dots + E(X_k^2) &\leq P \\ \Rightarrow \text{tr}(K_x) &\leq P. \end{aligned}$$

Since $Z \sim \mathcal{N}(0, K_z) \rightarrow H(z) = \frac{1}{2} \log(2\pi e |K_z|^n)$. The problem becomes

$$\begin{aligned} \max \quad I(X; Y) &= \frac{1}{2} \log \left(\frac{\det(K_z + K_x)}{\det(K_z)} \right) \\ \text{s.t.} \quad \text{tr}(K_x) &\leq P, \end{aligned}$$

Let $K_z = Q\Lambda_z Q^T$ be the eigenvalue decomposition, then $K_x = Q\Lambda_x Q^T$ is the eigenvalue decomposition of the optimal K_x , where $\lambda_{x_i} = (\mu - \lambda_{N_i})^+$. In other words, we design the the input signals to be correlated with the same eigenvectors as the noise correlation, and perform water-filling on the eigenvalues of K_z . This is also the idea in a MIMO channel if both the transmitter and receiver know the channel.

4 Bandlimited Channels

The bandlimited channel is the channel with spectrum $H(f) = 0$ if $|f| \geq W$. With sample rate $2W$, we could reconstruct the signal completely. For each sample, the power constrain becomes $\frac{P}{2W}$. Assume white noise with Power Spectrum Density(PSD) $\frac{N_0}{2}$, then the total noise power = N_0W , and noise power per sample is $\frac{N_0W}{2W} = \frac{N_0}{2}$

So the capacity per sample is

$$C_s = \frac{1}{2} \log \left(1 + \frac{P}{N_0W} \right).$$

Then the capacity for the bandlimited channel is

$$C = 2WC_s = W \log \left(1 + \frac{P}{N_0W} \right).$$

If the noise is not white but is a stationary Gaussian stochastic process with covariance matrix K_z , then we perform waterfilling on the eigenvalues of K_z . As the number of samples increases, the density of the eigenvalue of K_z tends to the power spectrum of the noise, and waterfilling translates to the spectral domain.